Systematic Rewiring in Associative Neural Networks with Small-World Architecture

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Abstract– **It is a known fact that a small amount of randomly rewired connections greatly improves the performance of associative neural network with regular architecture, still preserving its attractive features such as local connectivity and small total connection length. In this paper we propose the systematic way of connection rewiring which further improves the associative properties of the network using the same amount of rewiring.**

I. Introduction

An analytical model for the small-world social phenomenon [1] (also known as "six degrees of separation") was proposed in [2]. The system with small-world architecture is characterized by low average path length, as in a random network, meanwhile having high clustering coefficient, like regular lattices. Many networks in social, biological, technological systems are shown to exhibit these properties [1, 2].

Models of sparse (or diluted) associative neural networks based on small-world architecture have been studied intensively in recent years. In [3] authors gave statistical characterization of attractors depending on the value of network disorder. The comparison of networks with scale-free and small-world topologies is done in [4]. Authors of [5] showed that associative memory networks with small world architectures provide the same retrieval performance as randomly connected networks while using only a fraction of the total connection length. Instead of the Hebbian learning rule used in [3-5] authors of [6] use the better performing perceptron learning rule and investigate the influence of connections/weights symmetry on network's associative properties.

All models from [3-6] use the same rewiring procedure where the connections are changed in a random way [2]. We propose a different approach for rewiring, which is inspired by the weight selection algorithm [7].

It has been shown in [7] that a fully connected Hopfield network still performs well even after the removal of 80% of neuron connections with the least absolute values. The location of the remaining connections is apparently of great importance for the associative properties of the network and reflects some hidden interrelationships in stored data.

This approach was used in [8, 9] for setting the architecture of a sparse network. The resulting network had better associative recall properties than the network with the equal amount of randomly set connections.

The same idea is exploited for the proposed systematic rewiring. The location of connections with the largest and smallest absolute values in fully connected Hopfield network trained with the same data set is used to move connections during the rewiring.

II. The Associative Model

We consider a Hopfield-type sparse associative neural network, consisting of n neurons. Neuron j affects neuron i if and only if

$$j \in N_i \tag{1}$$

where $N_i \subset \{1,\dots,n\}$ is a subset of unique indices.

The network architecture is characterized by the density of connections, or connectivity:

$$\rho = \sum_{i=1}^{n} |N_i| \Big/ n^2 \tag{2}$$

and by the total connection length:

$$l = \sum_{i=1}^{n} \sum_{j=1}^{|N_i|} |i - N_i[j]| \tag{3}$$

There is no direct connection from a neuron to itself: .

The neuron input, or local field of the i-th neuron, is calculated as a weighted sum of net outputs:

(4)

where W is the (n×n) weight matrix of interneuron connections.

During the convergence process the neuron output at the next time step is obtained after applying some monotonous nonlinear activation function to the neuron input at the current time step:

(5)

In the given work as an activation function we use the sign function with codomain {-1, +1}. Therefore the network stores bipolar vectors with {-1, +1} components.

Neuron states can be updated synchronously or asynchronously. We use a synchronous update mode that favours parallel processing in a hardware implementation and offers better associative properties to the network.

III. Small-World Architecture

The construction of small-world architecture begins with a one-dimensional, n-node ring lattice, with each node linked to its $k/2$ nearest neighbours on each side:

(6)

Then some of the connections are rewired (in one of the following ways), that brings this system to an intermediate state between a regular lattice and disordered network.

A. Random Rewiring

In the random rewiring procedure [2], with a probability p, each connection is rewired to a randomly chosen node in the network. Self-connections and duplicate connections are forbidden.

B. Systematic Rewiring

Suppose we want the network to store a set of n-ary, bipolar training vectors

(7)

First we train a fully connected Hopfield network using this dataset and a projective learning rule [10]:

(8)

which results in weight matrix being equal to the projection matrix onto a subspace spanned on data vectors $\{\xi^p\}$.

Then we take a network with a regular lattice architecture described above and for each i-th neuron sort

– its existing connections in ascending order of absolute value of corresponding weight in W_{Full} ;
– its non-existing connections in descending order of absolute value of corresponding weight in W_{Full} .

For the i-th neuron we find the maximum number of its connections that are allowed to be rewired:

(9)

(i.e. we accept only such rewiring of a connection that would result in larger absolute value of corresponding element of W_{Full} than it was before the rewiring).

Then, for a given rewiring probability p, so as to get the expected number of rewired connections close to $p \cdot n \cdot k$ (as it would happen in case of random rewiring), we perform the rewiring in a following way:

(10)

where

(11)

(denotes integer part, or floor function).

Updated sets of neuron indices determine the new network structure after the systematic rewiring.

IV. Learning Algorithm

We deploy a two-phase algorithm. The first phase is setting small-world architecture of the network and the second one is assigning weights to the connections using the Pseudo Inverse learning rule (PI LR) [11].

For sufficiently connected networks ($\rho \sim m/n$) and non-singular training data $\{\xi^p\}$ PI LR is guaranteed to match its learning criterion – the equality of all aligned local fields (17) to 1. The algorithm works as follows.

To allow for structural restrictions imposed by the sparse architecture we introduce a selection operator that sparsifies the columns of a matrix:

(12)

Operator S^i retains only those columns of its matrix argument that correspond to indices contained in N_i.

Denoting the i-th row of the training data matrix as $\{\xi^p\}^i$, the weights of the i-th neuron are calculated as a solution of the following "fixed point" equation:

(13)

This solution can be found using matrix pseudo inversion operator:

(14)

V. Associative Properties

A. Attraction Radius

Let $m(\xi,\eta)$ be the overlap between two bipolar n-dimensional vectors:

(15)

If the network has associative properties with respect to the training set $\{\xi^p\}$ then the convergence process starting from a point η, such as its overlap with one of the stored vectors $m(\xi^i,\eta) = m_0 < 1$, finishes exactly at ξ^i.

To find minimum value of m_0 for each of the stored vectors ξ^i we started the convergence process from a random initial point η ($m(\xi^i,\eta) = m_0 \cong 0$). Then the overlap was incrementally increased (by a value of $1/n$) until the convergence process reaches ξ^i.

In our experiments we use the normalized attraction radius R [12], which is defined as

(16)

where m_0 is minimum overlap providing the convergence to the stored vector ξ^i, m_1 is a maximum overlap of ξ^i with the rest of the stored vectors, and the double average is taken over different sets of stored patterns and every pattern ξ^i in a particular dataset.

B. Kappa Estimation of Attractor Performance

It is always possible to estimate attraction radius experimentally. But it can be computationally expensive to obtain a reliable value of R for high dimensional networks. In the situation when one is not interested in quantitative estimation of attractor performance but rather wants to compare different networks, another approach can be used.

The aligned local field for the i-th neuron and bipolar vector ξ is a value:

(17)

The bipolar vector is a stable point for the network if and only if the aligned local fields of all neurons are positive.

An upward scaling of the weight matrix increases all aligned local fields, but obviously does not affect the associative properties of the network. Optimal performance can be achieved with the maximization of the aligned local fields with respect to the size of weights. It can be completed by maximizing the normalized stability measure [13]:

(18)

where W^i is the vector of weight coefficients of neuron i (the i-th row of W).

The minimum over all neurons and training patterns gives a measure of expected attractor performance (κ-*measure*):

(19)

VI. Numerical Simulations

A. Experimental Setup

For each value of rewiring probability we generated 10 datasets $\{\xi^p\}$. Each training set $\{\xi^p\}$ was composed of random data vectors with independent and equiprobable components {-1, +1}. The size of the data set is 10 and its dimension is 400 (n = 400, m = 10).

All network characteristics are averaged over these 10 datasets and averages and standard deviations are shown in graphs.

B. Experimental Results

We tested the network with neuron neighborhood size k = 20 (ρ = 0.05). For the network with systematic rewiring algorithm we plotted the fraction of actually rewired connections, and this value was exactly equal to the parameter p of the algorithm up to a point p = 0.8. For larger values of p the fraction of rewired connections was slightly suppressed by the limitation (11).

Fig.1 shows the dependence of normalized attraction radius R on rewiring parameter p. It can be seen that the systematic rewiring allows achieving of larger values of R using the same number of rewired connections. Moreover, in a range of $p \in [0.1, 0.75]$ it provides better associative recall then the network with completely random architecture (p = 1 in the random rewiring algorithm).

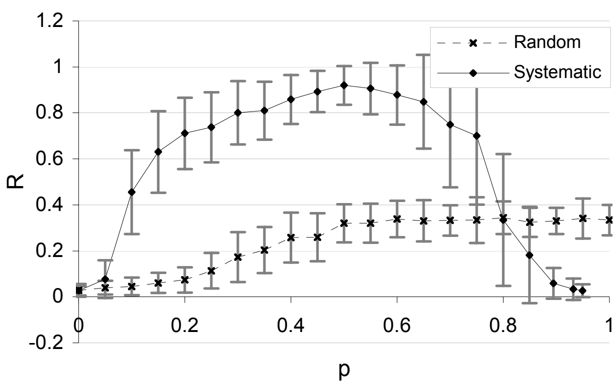

Fig. 1. Normalized attraction radius vs. rewiring parameter

The values of kappa estimation are shown in fig. 2. For each value of rewiring parameter p it gives an appropriate comparison of two rewiring approaches (similar to direct experimental comparison by means of attraction radius in fig. 1). But the evolution of κ-measure along with p for random rewiring algorithm does not reveal the observed improvement in network associative behaviour; therefore the usage of kappa estimation should always be backed up by the direct experimentation.

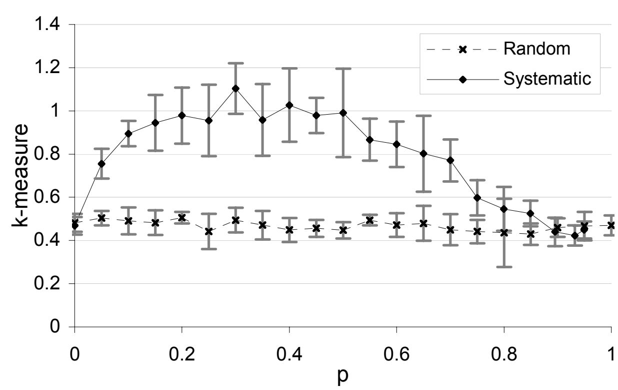

Fig. 2. k-measure vs. rewiring parameter.

Systematic rewiring algorithm shows an interesting behaviour for values of rewiring parameter $p > 0.5$. The network associative performance starts to deteriorate, approaching the state of virtually no recall ($R = 0.02$) at $p = 0.95$. This could be explained by looking at the average rank of the system of equations (13). The rank is averaged both over all neurons and all datasets used in testing. If (13) has an exact solution, than its rank is equal to $m = 10$ (number of stored patterns). This always holds true for the random rewiring scheme, while for the systematic rewiring lower system rank means the absence of an exact solution for the weights of some neurons. Such "faulty" neurons eventually cause the deterioration of network associative properties.

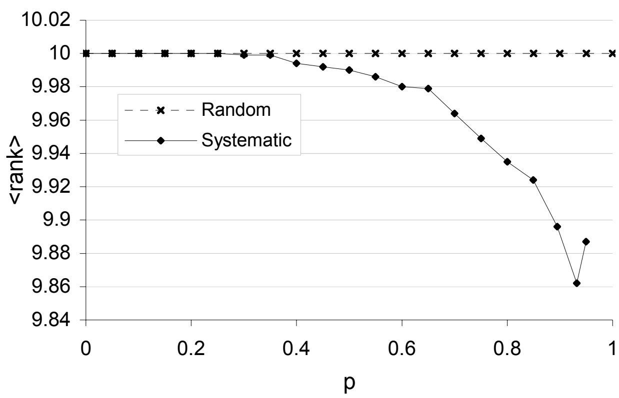

Fig. 3. Average system rank vs. rewiring parameter.

In [5] it has been pointed out that associative networks with small world-architecture allow achieving of the same associative properties compare to the networks with random architecture, meanwhile offering significant economy in total connection length. Fig. 4 shows that the systematic rewiring procedure preserves this attractive feature resulting just in a minor increase of total connection length.

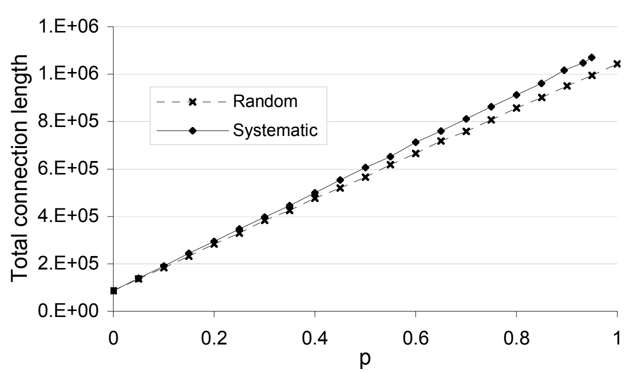

Fig. 4. Total connection length vs. rewiring parameter.

VII. Conclusions and Future Work

Experimental results show the advantages of systematic rewiring procedure. As a matter of fact, doing the rewiring using the weight selection algorithm, we are performing the selection of variables (neuron weights) of equation (13) using the information about the constraints they must satisfy. This approach is more optimal for the solvability of (13), providing a lower-norm solution (higher value of κ-measure) and, hence, better associative properties of the network.

In the context of the future work we plan to compare the combination of systematic rewiring with different learning rules. The usage of covariance matrix (correlation LR) for the weight selection instead of the projective one is also an attractive approach as it simplifies the rewiring algorithm.

Acknowledgement

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References

[1] S. Milgram, "The small world problem," *Psychology Today*, vol. 2, pp. 60–67, 1967.

[2] D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world' networks," *Nature*, vol. 393, pp. 440-442, 1998.

[3] L. G. Morelli, G. Abramson and M. N. Kuperman, "Associative memory on a small-world neural network," *The European Physical Journal B - Condensed Matter*, vol. 38, pp. 495-500, 2004.

[4] P. N. McGraw and M. Menzinger, "Topology and computational performance of attractor neural networks," *Physical Review E*, vol. 68, 2003.

[5] J. W. Bohland and A. A. Minai, "Efficient Associative Memory Using Small-World Architecture," *Neurocomputing*, vol. 38-40, pp.489-496, 2001.

[6] N. Davey, B. Christianson and R. Adams, "High Capacity Associative Memories and Small World Networks," *Proceedings of IEEE Int. Joint Conf. on Neural Networks*, Budapest, Hungary, 25-29 July, 2004.

[7] A.S. Sitchov, "Weight selection in neural networks with pseudoinverse learning rule," (in Russian), *Mathematical Machines and Systems,* vol.2, pp. 25-30, 1998.

[8] O. K. Dekhtyarenko, A. M. Reznik and A. S. Sitchov "Associative Cellular Neural Networks with Adaptive Architecture," *Proceedings of IEEE Workshop on Cellular Neural Networks and their Applications*, Budapest, Hungary, 22-24 July, 2004.

[9] O. Dekhtyarenko, V. Tereshko, and C. Fyfe, "Phase transition in sparse associative neural networks," accepted for *European Symposium on Artificial Neural Networks*, Bruges, Belgium, 27-29 April, 2005.

[10] L. Personnaz, I. Guyon, G. Dreyfus: "Collective computational properties of neural networks: New learning mechanisms," *Physical Review A*, vol. 34(5), pp. 4217-4228, 1986.

[11] M. Brucoli, L. Carnimeo and G. Grassi, "Discrete-time cellular neural networks for associative memories with learning and forgetting capabilities", *IEEE Transactions on Circuits and Systems*, vol. 42(7), pp. 396-399, 1995.

[12] I. Kanter, and H. Sompolinsky, "Associative Recall of Memory Without Errors," *Physical Review A*, vol. 35, pp. 380-392, 1987.

[13] T.B. Kepler and L.F. Abbot, "Domains of attraction in neural networks," *Journal de Physique*, vol. 49, pp. 1657-1662, 1988.